III. "An Experimental Investigation of the Circumstances which Determine whether the Motion of Water shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels." By OSBORNE REYNOLDS, F.R.S. Received March 7, 1883.

(Abstract.)

1. *Objects and Results of the Investigation.*—The results of this investigation have both a practical and a philosophical aspect.

In their practical aspect they relate to the *laws of resistance to the motion of water in pipes*, which appears in a new form, the law for all velocities and all diameters being represented by an equation of two terms.

In their philosophical aspect these results relate to the fundamental principles of fluid motion; inasmuch as they afford for the case of pipes a definite verification of two principles, which are *that the general character of the motion of fluids in contact with solid surfaces depends on the relation* (1) *between the dimensions of the space occupied by the fluid and a linear physical constant of the fluid;* (2) *between the velocity and a physical velocity constant of the fluid.*

The results as viewed in their philosophical aspect were the primary object of the investigation.

As regards the practical aspect of the results, it is not necessary to say anything by way of introduction; but in order to render the philosophical scope and purpose of the investigation intelligible, it is necessary to describe shortly the line of reasoning which determined the order of investigation.

2. *The Leading Features of the Motion of Actual Fluids.*—Although in most ways the exact manner in which water moves is difficult to perceive, and still more difficult to define, as are also the forces attending such motion, certain general features both of the forces and motions stand prominently forth as if to invite or defy theoretical treatment.

The relations between the resistance encountered by, and the velocity of a solid body moving steadily through a fluid in which it is completely immersed, or of water moving through a tube, present themselves mostly in one or other of two simple forms. The resistance is generally proportional to the square of the velocity, and when this is not the case it takes a simpler form, and is proportional to the velocity.

Again, the internal motion of water assumes one or other of two broadly distinguishable forms—either the elements of the fluid follow one another along lines of motion which lead in the most direct manner to their destination, or they eddy about in sinuous paths, the most indirect possible.

The transparency or uniform opacity of most fluids renders it impossible to see the internal motion, so that, broadly distinct as are the two classes (direct and sinuous) of motion, their existence would not have been perceived, were it not that the surface of water, where otherwise undisturbed, indicates the nature of the motion beneath. A clean surface of moving water has two appearances, the one like that of *plate glass* in which objects are reflected without distortion; the other like that of *sheet glass*, in which the reflected objects appear crumpled up and grimacing. These two characters of surface correspond to the two characters of motion. This may be shown by adding a few streaks of highly coloured water to the clear moving water. Then, although the coloured streaks may at first be irregular they will, if there are no eddies, soon be drawn out into even colour bands; whereas if there are eddies, they will be curled and whirled about in the manner so familiar with smoke.

3. *Connexion between the Leading Features of Fluid Motion.*— These leading features of fluid motion are well known, and are supposed to be more or less connected, but it does not appear that hitherto any very determined efforts have been made to trace a definite connexion between them, or to trace the characteristics of the circumstances under which they are usually presented.

Certain circumstances have been definitely associated with the particular laws of force. Resistance as the square of the velocity is associated with motion in tubes of more than capillary dimensions, and with the motion of the bodies through the water at more than insensibly small velocities, while resistance as the velocity is associated with capillary tubes and small velocities.

The equations of hydrodynamics, although they are applicable to *direct motion*, *i.e.*, without eddies, and show that then the resistance is as the velocity, have hitherto thrown no light on the circumstances on which such motion depends. And although of late years these equations have been applied to the theory of the eddy, they have not been in the least applied to the motion of water, which is a mass of eddies, *i.e.*, in *sinuous motion*, nor have they yielded a clue to the cause of resistance varying as the square of the velocity. Thus, while as applied to waves and the motion of water in capillary tubes the theoretical results agree with the experimental, the theory of hydrodynamics has so far failed to afford the slightest hint why it should explain these phenomena, and signally failed to explain the law of resistance encountered by large bodies moving at sensibly high velocities through water, or that of water in sensibly large pipes.

This accidental fitness of the theory to explain certain of the phenomena, while entirely failing to explain others, affords strong presumption that there are some fundamental principles of fluid motion of which due account has not been taken in the theory; and

several years ago it seemed to me that a careful examination as to the connexion between these four leading features, together with the circumstances on which they severally depend, was the most likely means of finding the clue to the principles overlooked.

4. *Space and Velocity.*—The definite association of resistance as the square of the velocity with sensibly large tubes and high velocities, and of resistance as the velocity with capillary tubes and slow velocities, seemed to be evidence of the very general and important influence of some properties of fluids not recognised in the theory of hydrodynamics.

As there is no such thing as absolute space or absolute time recognised in mechanical philosophy, to suppose that the character of motion of fluids in any way depended on absolute size or absolute velocity would be to suppose such motion outside the pale of the laws of motion. If, then, fluids, in their motions, are subject to these laws, what appears to be the dependence of the character of the motion on the absolute size of the tube and on the absolute velocity of the immersed body must in reality be a dependence on the size of the tube as compared with the size of some other object, and on the velocity of the body as compared with some other velocity. What is the standard object and what the standard velocity which come into comparison with the size of the tube and the velocity of an immersed body, are questions to which the answers were not obvious. Answers, however, were found in the discovery of a circumstance on which sinuous motion depends.

5. *The Effect of Viscosity on the Character of Fluid Motion.*—The small evidence which clear water shows as to the existence of internal eddies, not less than the difficulty of estimating the viscous nature of the fluid, appears to have hitherto obscured the very important circumstance that *the more viscous a fluid is the less prone is it to eddying or sinuous motion.* To express this definitely, if μ is the viscosity and ρ the density of the fluid, for water $\frac{\mu}{\rho}$ diminishes rapidly as the temperature rises; thus at 5° C. $\frac{\mu}{\rho}$ is double what it is at 45° C. What I observed was that the tendency of water to eddy becomes much greater as the temperature rises.

Hence, connecting the change in the law of resistance with the birth and development of eddies, this discovery limited further search for the standard distance and standard velocity to the physical properties of the fluid.

To follow the line of this search would be to enter upon a molecular theory of liquids, and this is beyond my present purpose. It is sufficient here to notice the well known fact that—

$$\frac{\mu}{\rho}$$

is a quantity of the nature of the product of a distance and a velocity; and to point out that the establishment of a dependence of the character of fluid motion on a relation between the linear size of the space, the velocity of the fluid, and $\frac{\mu}{\rho}$, would be equivalent to establishing the existence of two physical constants, one a distance and the other a velocity or a time, as amongst the properties of fluids. Using the term dimension as implying measures of time as well as space, these constants may well be called dimensional properties or fluids. Similar constants are already recognised; thus the velocity of sound is such a velocity constant, and the mean paths of gaseous molecules, or the mean range, are such linear constants.

It is always difficult to trace the dependence of one idea on another; but it may be noticed that no idea of dimensional properties, as indicated by the dependence of the character of motion on the size of the tube and the velocity of the fluid, occurred to me until after the completion of my investigation on the transpiration of gases, in which was established the dependence of the law of transpiration on the relation between the size of the channel and the *mean range* of the gaseous molecules.

6. *Evidence of Dimensional Properties in the Equations of Motion.*—The equations of motion had been subjected to such close scrutiny, particularly by Professor Stokes, that there was small chance of discovering anything new or faulty in them. It seemed to me possible, however, that they might contain evidence which had been overlooked, of the dependence of the character of motion on a relation between the dimensional properties and the external circumstances of motion. Such evidence, not only of a connexion, but of a definite connexion, was found, and this without integration.

If the motion be supposed to depend on a single velocity parameter U—say the mean velocity along a tube—and on a single linear parameter c, say the radius of the tube; then, having in the usual manner eliminated the pressure from the equations, there remain two types of terms in one of which—

$$\frac{U^2}{c^3}$$

is a factor, and in the other—

$$\frac{\mu U}{\rho c^4}$$

is a factor. So that the relative values of these terms vary respectively as U and—

$$\frac{\mu}{c\rho}.$$

This is a definite relation of the exact kind for which I was in

search. Of course, without integration the equations only gave the relation, without showing at all in what way the motion might depend upon it. It seemed, however, to be certain, if the eddies were owing to one particular cause, that integration would show the birth of eddies to depend upon some definite value of—

$$\frac{c\rho U}{\mu}$$

7. *The Cause of Eddies.*—There appeared to be two possible causes for the change of direct motion into sinuous. These are best discussed in the language of hydrodynamics; but as the results of this investigation relate to both these causes, which, although the distinction is subtile, are fundamentally distinct and lead to distinct results, it is necessary that they should be indicated.

The general cause of the change from steady to eddying motion was, in 1843, pointed out by Professor Stokes as being that, under certain circumstances, the steady motion becomes unstable, so that an indefinitely small disturbance may lead to a change to sinuous motion. Both the causes above referred to are of this kind, and yet they are distinct; the distinction lying in the part taken in the instability by viscosity. If we imagine a fluid free from viscosity and absolutely free to glide over solid surfaces, then comparing such a fluid with a viscous fluid in exactly the same motion—

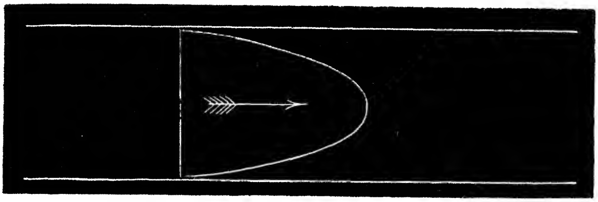
(1.) The frictionless fluid might be unstable and the viscous stable. Under these circumstances the cause of eddies is the instability as a perfect fluid, the effect of viscosity being in the direction of stability.

(2.) The frictionless fluid might be stable and the viscous fluid unstable; under which circumstances the cause of instability would be the viscosity.

It was clear to me that the conclusion I had drawn from the equations of motion immediately related only to the first cause. Nor could I then perceive any possible way in which instability could result from viscosity. All the same I felt a certain amount of uncertainty in assuming the first cause of instability to be general. This uncertainty was the result of various considerations, but particularly from my having observed that eddies apparently come on in very different ways, according to a very definite circumstance of motion, which may be illustrated.

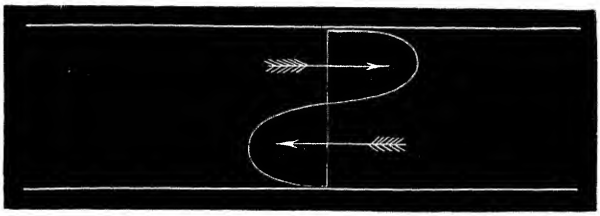
When in a channel the water is all moving in the same direction, the velocity being greatest in the middle and diminishing to zero at the sides, as indicated by the curve in fig. 1, eddies showed themselves reluctantly and irregularly; whereas when the water on one side of the channel was moving in the opposite direction to that on

FIG. 1.



the other, as shown by the curve in fig. 2, eddies appeared in the middle regularly and readily.

FIG. 2.



8. *Methods of Investigation.*—There appeared to be two ways of proceeding, the one theoretical, the other practical.

The theoretical method involved the integration of equations for unsteady motion in a way that had not then been accomplished, and which, considering the general intractability of the equations, was not promising.

The practical method was to test the relation between U, $\frac{\mu}{\rho}$, and c; this, owing to the simple and definite form of the law, seemed to offer, at all events in the first place, a far more promising field of research.

The law of motion in a straight smooth tube offered the simplest possible circumstances and the most crucial test.

The existing experimental knowledge of the resistance of water in tubes, although very extensive, was in one important respect incomplete. The previous experiments might be divided into two classes—(1) those made under circumstances in which the law of resistance was as the square of the velocity, and (2) those made under circumstances in which the resistance varied as the velocity. There had not apparently been any attempt made to determine the exact circumstances under which the change of law took place.

Again, although it had been definitely pointed out that eddies would explain the resistance as the square of the velocity, it did not appear that any definite experimental evidence of the existence of eddies in parallel tubes had been obtained, and much less was there

any evidence as to whether the birth of eddies was simultaneous with the change in the law of resistance.

These open points may be best expressed in the form of queries to which the answers anticipated were in the affirmative.

(1.) What was the exact relation between the diameters of the pipes and the velocities of the water at which the law of resistance changed; was it at a certain value of

$$c\mathrm{U}?$$

(2.) Did this change depend on the temperature, *i.e.*, the viscosity of water; was it at a certain value of

$$\frac{\mathrm{U}}{\mu}?$$

(3.) Were there eddies in parallel tubes?

(4.) Did steady motion hold up to a critical value and then eddies come in?

(5.) Did the eddies come in at a certain value of

$$\frac{\rho c\mathrm{U}}{\mu}?$$

(6.) Did the eddies first make their appearance as small, and then increase gradually with the velocity, or did they come in suddenly?

The bearing of the last query may not be obvious; but, as will appear in the sequel, its importance was such that in spite of satisfactory answers to all the other queries, a negative answer to this in respect of one particular class of motions led to the reconsideration of the supposed cause of instability and eventually to the discovery of instability caused by fluid friction.

The queries as they are put suggest two methods of experimenting:—

(1.) Measuring the resistances and velocities for different diameters, and with different temperatures of water.

(2.) Visual observation as to the appearance of eddies during the flow of water along tubes or open channels.

Both these methods have been adopted, but as the question relating to eddies had been the least studied the second method was the first adopted.

9. *Experiments by Visual Observations.*—The most important of these experiments related to water moving in one direction along glass tubes. Besides these, however, experiments on fluids flowing in opposite directions in the same tube were made; also a third class of experiments which related to motion in a flat channel of indefinite breadth.

These last-mentioned experiments resulted from an incidental observation during some experiments made in 1876 as to the effect

of oil to prevent wind waves. As the result of this observation had no small influence in directing the course of this investigation, it may be well to describe it first.

10. *Eddies caused by the Wind beneath the Oiled Surface of Water.*—A few drops of oil on the windward side of a pond during a stiff breeze having spread over the pond and completely calmed the surface as regards waves, the sheet of oil, if it may be so called, was observed to drift before the wind, and it was then particularly noticed that close to, and at a considerable distance from, the windward edge, the surface presented the appearance of *plate glass;* further from the edge the surface presented that wavering appearance which has already been likened to that of sheet glass, which appearance was at the time noted as showing the existence of eddies beneath the surface.

Subsequent observation confirmed this first view. At a sufficient distance from the windward edge of an oil-calmed surface there are always eddies beneath the surface even when the wind is light. But the distance from the edge increases rapidly as the force of the wind diminishes, so that at a limited distance (10 or 20 feet) the eddies will come and go with the wind.

Without oil I was unable to perceive any indication of eddies. At first I thought that the waves might prevent their appearance even if they were there, but by careful observation I convinced myself that they were not there. It is not necessary to discuss these results here, although, as will appear, they have a very important bearing on the cause of instability.

11. *Experiments by Means of Colour Bands in Glass Tubes.*—These were undertaken early in 1880; the final experiments were made on three tubes, Nos. 1, 2, and 3.

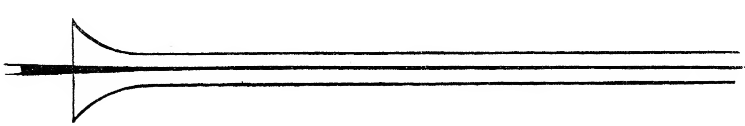
The diameters of these were nearly 1-inch, $\frac{1}{2}$-inch, and $\frac{1}{4}$-inch. They were all about 4 feet 6 inches long, and fitted with trumpet mouthpieces, so that water might enter without disturbance.

The water was drawn through the tubes out of a large glass tank in which the tubes were immersed, arrangements being made so that a streak or streaks of highly coloured water entered the tubes with the clear water.

The general results were as follows:—

(1.) When the velocities were sufficiently low, the streak of colour extended in a beautiful straight line through the tube, fig. 3.

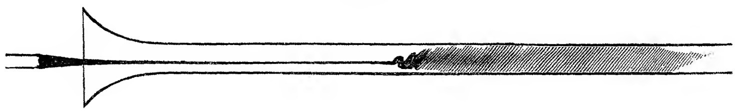
FIG. 3.



(2.) If the water in the tank had not quite settled to rest, at sufficiently low velocities the streak would shift about the tube, but there was no appearance of sinuosity.

(3.) As the velocity was increased by small stages at some point in the tube always at a considerable distance from the trumpet or intake, the colour band would all at once mix up with the surrounding water, and fill the rest of the tube with a mass of coloured water, as in fig. 4.

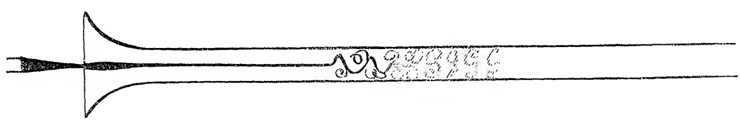
Fig. 4.



Any increase in the velocity caused the point of breakdown to approach the trumpet, but with no velocities that were tried did it reach this.

On viewing the tube by the light of an electric spark, the mass of colours resolved itself into a mass of more or less distinct curls showing eddies, as in fig. 5.

Fig. 5.



The experiments thus seemed to settle questions 3 and 4 in the affirmative—the existence of eddies and a critical velocity.

They also settled in the negative, question 6 as to the eddies coming in gradually after the critical velocity was reached.

In order to obtain an answer to question 5 as to the law of the critical velocity, the diameters of the tubes were carefully measured, also the temperature of the water and the rate of discharge.

(4.) It was then found that with water at a constant temperature and the tank as still as could by any means be brought about, the critical velocities at which the eddies showed themselves were exactly in the inverse ratios of the diameters of the tubes.

(5.) That in all the tubes the critical velocity diminished as the temperature increased, the range being from 5° C. to 22° C. and the law of this diminution, so far as could be determined, was in accordance with Poiseuille's experiment.

Taking T to express degrees Centigrade, then by Poiseuille's experiments—

$$\frac{\mu}{\rho} \propto \text{P} = 1 + 0{\cdot}0336\,\text{T} + 0{\cdot}00221\,\text{T}^2,$$

Taking a metre as the unit, U_s the critical velocity, and D the diameter of the tube, the law of the critical point is completely expressed by the formula

$$U_s = \frac{1}{B_s} \frac{P}{D}$$

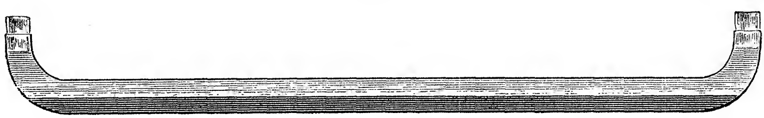
where $$B_s = 43\cdot7$$

This is a complete answer to question 5.

During the experiments many things were noticed which cannot be mentioned here, but two circumstances should be mentioned as emphasizing the negative answer to question 6. In the first place, the critical velocity was much higher than had been expected in pipes of such magnitude, resistance varying as the square of the velocity had been found at very much smaller velocities than those at which the eddies appeared when the water in the tank was steady. And in the second place it was observed that the critical velocity was very sensitive to disturbance in the water before entering the tubes, and it was only by the greatest care as to the uniformity of the temperature of the tank and the stillness of the water that consistent results were obtained. This showed that the steady motion was unstable for large disturbances long before the critical velocity was reached, a fact which agreed with the full blown manner in which the eddies appeared.

12. *Experiments with two Streams in Opposite Directions in the same Tube.*—A glass tube 5 feet long and 1·2 inch in diameter, having its ends slightly bent up as shown in fig 6, was half filled with bisulphide

FIG. 6.



of carbon, and then filled up with water and both ends corked. The bisulphide was chosen as being a limpid liquid, but little heavier than water and completely insoluble, the surface between the two liquids being clearly distinguishable. When the tube was placed in a horizontal direction, the weight of the bisulphide caused it to spread along the lower half of the tube, and the surface of separation of the two liquids extended along the axis of the tube.

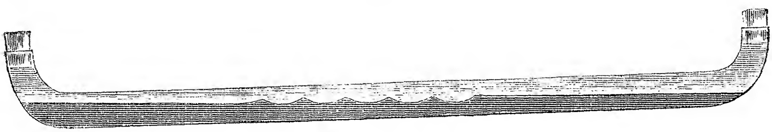
On one end of the tube being slightly raised the water would flow to the upper end, and the bisulphide fall to the lower, causing opposite currents along the upper and lower halves of the tube, while in the middle of the tube the level of the surface of separation remained unaltered.

The particular purpose of this investigation was to ascertain whether there was a critical velocity at which waves or sinuosities would show themselves in the surface of separation. It proved a very pretty experiment and completely answered its purpose.

When one end was raised quickly by a definite amount, the opposite velocities of the two liquids, which were greatest in the middle of the tube, attained a certain maximum value depending on the inclination given to the tube. When this was small no signs of eddies or sinuosities showed themselves, but at a certain definite inclination waves (nearly stationary) showed themselves, presenting all the appearance of wind waves.

These waves first made their appearance as very small waves of equal lengths, the length being comparable to the diameter of the tube.

Fig. 7.



When by increasing the rise, the velocities of flow were increased, the waves kept the same length, but became higher, and when the rise was sufficient, the waves would curl and break, the one fluid winding itself into the other in regular eddies.

Whatever might be the cause, a skin formed slowly between the bisulphide and the water, and this skin produced similar effects to that of oil on water, the results mentioned are those which were obtained before the skin showed itself. When the skin first came on regular waves ceased to form, and in their place the surface was disturbed as if by irregular eddies above and below, just as in the case of the oiled surface of water.

The experiment was not adapted to afford a definite measure of the velocities at which the various phenomena occurred, but it was obvious that the critical velocity at which the waves first appeared, was many times smaller than the critical velocity in a tube of the same size when the motion was in one direction only. It was also clear that the critical velocity was nearly if not quite independent of any existing disturbance in the liquids. So that this experiment shows—

(1.) That there is a critical velocity in the case of opposite flow, at which direct motion becomes unstable.

(2.) That the instability came on gradually and did not depend on the magnitude of the disturbances, or in other words, that for this class of motion question 6 must be answered in the affirmative.

It thus appeared that there was some difference in the cause of instability in the two motions.

13. *Further Study of the Equations of Motion.*—Having now definite data to guide me, I was anxious to obtain a fuller explanation of these results from the equation of motion. I still saw only one way open to account for the instability, namely, by assuming the instability of a frictionless fluid to be general.

Having found a method of integrating the equations as far as to show whether any particular form of steady motion is stable for a small disturbance, I applied this method to the case of parallel flow in a frictionless fluid. The results which I obtained at once were, that flow in one direction was stable, flow in opposite directions unstable. This was not what I was looking for, and I spent much time in trying to find a way out of it, but whatever objections my method of integration may be open to, I could make nothing less of it.

It was not until the end of 1882 that I abandomed further attempts with a frictionless fluid and attempted by the same method the integration of a viscous fluid. This change was in consequence of a discovery that in previously considering the effect of viscosity I had omitted to take fully into account the boundary conditions which resulted from the friction between the fluid and the solid boundary.

On taking these boundary conditions into account, it appeared that although the tendency of viscosity through the fluid is to render direct or steady motion stable, yet owing to the boundary condition resulting from the friction at the solid surface, the motion of the fluid irrespective of viscosity would be unstable. Of course this cannot be rendered intelligible without going into mathematics. But what I want to point out is that this instability, as shown by the integration of the equations of motion, depends on exactly the same relation

$$U \propto \frac{\mu}{\rho c}$$

as that previously found.

This explained all the practical anomalies, and particularly the absence of eddies below a pure surface of water exposed to the wind; for in this case, the surface being free, the boundary condition was absent, whereas the film of oil by its tangential stiffness introduced this condition. This circumstance alone seemed a sufficient verification of the theoretical conclusion.

But there was also the sudden way in which eddies came into existence in the experiments with the colour band, and the effect of disturbances to lower the critical velocity. These were also explained, for as long as the motion was steady the instability depended upon the boundary action alone, but once eddies introduced the stability would be broken down.

It thus appeared that the meaning of the experimental results had

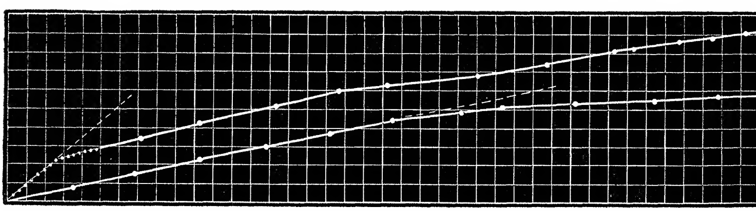
been ascertained, and the relation between the four leading features and the circumstances on which they depend traced, for the case of water in parallel flow. But as it appeared that the critical velocity in the case of motion in one direction did not depend on the cause of instability with a view to which it was investigated, it followed that there must be another critical velocity which would be the velocity at which previously existing eddies would die out, and the motion become steady as the water proceeded along the tube. This conclusion has been verified.

14. *Results of Experiments on the Law of Resistance in Tubes.*—The existence of the critical velocity described in the previous article could only be tested by allowing water in a high state of disturbance to enter a tube, and after flowing a sufficient distance for the eddies to die out, if they were going to die out, to test the motion. As it seemed impossible to apply the method of colour bands, the test applied was that of the law of resistance as indicated in questions (1) and (2) in § 8. The result was very happy. Two straight lead pipes, No. 4 and No. 5, each 16 feet long, and having diameters of a quarter and half inch respectively, were used.

The water was allowed to flow through rather more than 10 feet before coming to the first gauge-hole, the second gauge-hole being 5 feet further along the pipe.

The results were very definite, and are partly shown in fig. 8.

Fig. 8.



(1.) At the lower velocities the pressure was proportional to the velocity, and the velocities at which a deviation from this law first occurred were in the exact inverse ratio of the diameters of the pipes.

(2.) Up to these critical velocities the discharges from the pipes agreed exactly with those given by Poiseuille's formula for capillary tubes.

(3.) For some little distance after passing the critical velocity no very simple relations appeared to hold between the pressures and velocities; but by the time the velocity reached 1·3 (critical velocity) the relation became again simple. The pressure did not vary as the square of the velocity, but as 1·722 power of the velocity; this law

held in both tubes, and through velocities ranging from 1 to 50, where it showed no signs of breaking down.

(4.) The most striking result was that not only at the critical velocity, but throughout the entire motion the laws of resistance exactly corresponded for velocities in the ratio of

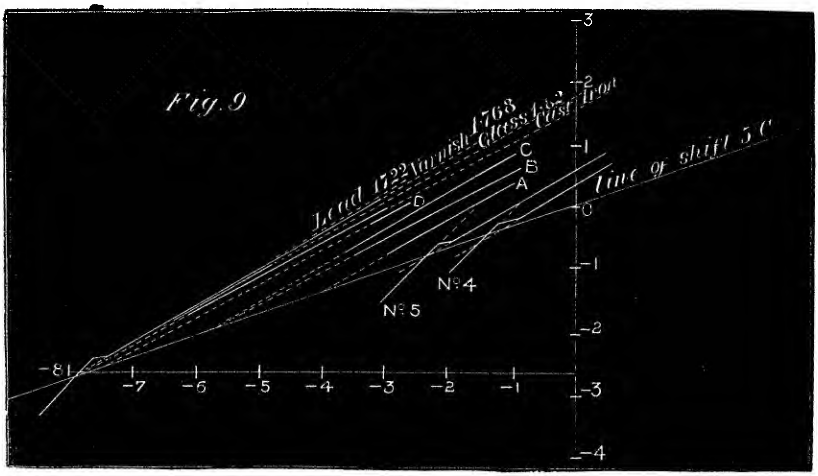
$$\frac{\mu}{\rho c}.$$

This last result was brought out in the most striking manner on reducing the results by the graphic method of logarithmic homologues as described in my paper on Thermal Transpiration.

Calling the resistance per unit of length as measured in the weight of cubic units of water i, and the velocity v, log i is taken for abscissa, and log v for ordinate, and the curve plotted.

In this way the experimental results for each tube are represented as a curve; these curves, which are shown as far as the small scale will admit in fig. 9, present exactly the same shape, and only differ in position.

FIG. 9.



Pipe.	Diameter.
	m.
No. 4, Lead........	0·00615
„ 5, „	0·0127
A, Glass	0·0496
B, Cast iron	0·188
D, „	0·5
C, Varnish	0·196

Either of the curves may be brought into exact coincidence with the other by a rectangular shift, and the horizontal shifts are given

by the difference of the logarithm of $\frac{D^3}{2}$ for the two tubes, the vertical shifts by the difference of the logarithm of

$$\frac{D}{\mu}.$$

The temperatures at which the experiment had been made were nearly the same, but not quite, so that the effect of the variations of μ showed themselves.

15. *Comparison with Darcy's Experiments.*—The definiteness of these results, their agreement with Poiseuille's law, and the new form which they more than indicated for the law of resistance above the critical velocity, led me to compare them with the well-known experiments of Darcy on pipes ranging from 0·014 to 0·5 metre. Taking no notice of the empirical laws by which Darcy had endeavoured to represent his results, I had the logarithmic homologues plotted from his published experiments. If my law was general then these log curves, together with mine, should all shift into coincidence if each were shifted horizontally through

$$\frac{D^3}{P^2}$$

and vertically through—

$$\frac{D}{P}.$$

In calculating these shifts there were some doubtful points. Darcy's pipes were not uniform between the gauge points, the sections varying as much as 20 per cent., and the temperature was only casually given. These matters rendered a close agreement unlikely; it was rather a question of seeing if there was any systematic disagreement. When the curves came to be shifted the agreement was remarkable; in only one respect was there any systematic disagreement, and this only raised another point; it was only in the slopes of the higher portions of the curves. In both my tubes the slopes were as 1·722 to 1; in Darcy's they varied according to the nature of the material, from the lead pipes, which were the same as mine, to 1·92 to 1 with the cast iron. This seems to show that the nature of the surface of the pipe has an effect on the law of resistance above the critical velocity.

16. *The Critical Velocities.*—All the experiments agreed in giving

$$v_c = \frac{1}{278}\frac{P}{D}$$

as the critical velocity, to which correspond as the critical pressure

$$i_c = \frac{1}{47700000}\frac{P^2}{D^3},$$

the units being metres and degrees Centigrade. It will be observed that this value is much less than the critical velocity at which steady motion broke down.

17. *General Law of Resistance.*—The log homologues all consist of two straight branches, the lower branch inclined at 45°, and the upper one at n horizontal to 1 vertical, except for the small distance beyond the critical velocity these branches constitute the curves. These two branches meet in a point o on the curve at a definite distance below the critical pressure, so that, ignoring the small portion of the curve above the point before it again coincides with the upper branch, the logarithmic homologues give for the law of resistance for all pipes and all velocities—

$$\mathrm{A}\frac{\mathrm{D}^3}{\mathrm{P}^2}i=\left(\mathrm{B}\frac{\mathrm{D}}{\mathrm{P}}v\right)^n,$$

where n has the value unity as long as either member is below unity, and then takes the value of the slope n to 1 for the particular surface of the pipe.

If the units are metres and degrees Centigrade—

$$\mathrm{A}=67{,}700{,}000,$$
$$\mathrm{B}=398,$$
$$\mathrm{P}=1+0\cdot0336\,\mathrm{T}+0\cdot000221\,\mathrm{T}^2.$$

This equation then, excluding the region immediately about the critical velocity, gives the law of resistance in Poiseuille's tubes, those of the present investigation, and Darcy's, the range of diameters being

from 0·000013 (Poiseuille, 1843),

to 0·5 (Darcy, 1857);

and the range of velocities—

from 0·0026 } metres per sec., 1883.
to 7 }

This algebraical formula shows that the experiments entirely accord with the theoretical conclusions. The empirical constants are A, B, P, and n; the first three relate solely to the dimensional properties of the fluid which enter into the viscosity, and it seems probable that the last relates to the properties of the surface of the pipe.

Much of the success of the experiments is due to the care and skill of Mr. Foster of Owens College, who has constructed the apparatus and assisted me in making the experiments.

The Society then adjourned over the Easter Recess to Thursday, April 5th.

FIG. 1.

FIG. 2.

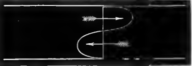


Fig. 4.

Fig. 5.

Fig. 6.

FIG. 7.

FIG. 8.

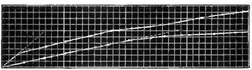
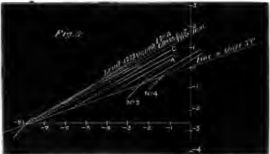


Fig. 9.



Pipe.	Diameter.
	m
No. 4, Lead.......	0·00615
„ 5, „	0·0127
A, Glass	0·0496
B, Cast iron	0·188
D, „	0·5
C, Varnish	0·196